Flow properties of differential equations driven by fractional Brownian motion

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Abstract

We prove that solutions of stochastic differential equations driven by fractional Brownian motion for $H > 1/2$ define flows of homeomorphisms on $\mathbb{R}^d$.

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1 Introduction

Suppose that $B^H = \{B_t^H, t \geq 0\}$ is an m-dimensional *fractional Brownian motion* with Hurst parameter $H \in (0,1)$, defined in a complete probability space $(\Omega, \mathcal{F}, P)$. That is, the components $B^{H,j}$, $j = 1, \ldots, m$, are independent zero mean Gaussian processes with the covariance function

$$R_H(t,s) = \frac{1}{2}\left(t^{2H} + s^{2s} - |t-s|^{2H}\right). \tag{1.1}$$

For $H = \frac{1}{2}$, the process B^H is an m-dimensional ordinary Brownian motion. On the other hand, from (1.1) it follows that

$$E(\left|B_t^{H,j} - B_s^{H,j}\right|^2) = |t-s|^{2H}.$$

As a consequence, the processes $B^{H,j}$ have stationary increments, and for any $\alpha < H$ we can select versions with Hölder continuous trajectories of order α on a compact interval $[0,T]$.

This process was first studied by Kolmogorov in [7] and later by Mandelbrot and Van Ness in [11], where a stochastic integral representation in terms of an ordinary Brownian motion was established.

The fractional Brownian motion has the following *self-similar* property: For any constant $a > 0$, the processes $\left\{a^{-H}B_{at}^H, t \geq 0\right\}$ and $\left\{B_t^H, t \geq 0\right\}$ have the same distribution. For $H = \frac{1}{2}$ the process B^H has independent

increments, but for $H \neq \frac{1}{2}$, this property is no longer true. In particular, if $H > \frac{1}{2}$, the fractional Brownian motion has the long range dependence property, which means that for each $j = 1, \ldots, m$

$$\sum_{n=1}^{\infty} \mathrm{Corr}(B_{n+1}^{H,j} - B_n^{H,j}, B_1^{H,j}) = \infty.$$

The self-similar and long range dependence properties make the fractional Brownian motion a convenient model for some input noises in a variety of topics from finance to telecommunication networks, where the Markov property is not required. This fact has motivated the recent development of the stochastic calculus with respect to the fractional Brownian motion. We refer to [12] for a survey of this topic.

In this paper we are interested in stochastic differential equations on $\mathbb{R}^d$ driven by a multi-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, that is, equations of the form

$$X_t^i = X_0^i + \sum_{j=1}^{m} \int_0^t \sigma^{i,j}(s, X_s) dB_s^{H,j} + \int_0^t b^i(s, X_s) ds, \tag{1.2}$$

$i = 1, \ldots, d$. The stochastic integral appearing in (1.2) is a path-wise Riemann-Stieltjes integral. In fact, under suitable conditions on σ, the processes $\sigma(s, X_s)$ and B_s^H have trajectories which are Hölder continuous of order strictly larger than $\frac{1}{2}$ and we can use the approach introduced by Young in [17]. A first result on the existence and uniqueness of a solution for this kind of equations was obtained by Lyons in [8], using the notion of p-variation. On the other hand, the theory of rough path analysis introduced by Lyons in [8] (see also the monograph by Lyons and Qian [10]), has allowed Coutin and Qian [3] to establish the existence of strong solutions and a Wong-Zakai type approximation limit for the stochastic differential equations of the form (1.2) driven by a fractional Brownian motion with parameter $H > \frac{1}{4}$. In [3] sufficient conditions on the vector fields b and σ are given to ensure existence and uniqueness of the solution of (1.2) even when vector fields do not commute.

In [18] Zähle has introduced a generalized Stieltjes integral using the techniques of fractional calculus. This integral is expressed in terms of fractional derivative operators and it coincides with the Riemann-Stieltjes integral $\int_a^b f dg$, when the functions f and g are Hölder continuous of orders λ and μ, respectively and $\lambda + \mu > 1$. Using this formula for the Riemann-Stieltjes integral, Nualart and Răşcanu have obtained in [14] the existence of

a unique solution for the stochastic differential equations (1.2) under general conditions on the coefficients.

Later on, Nualart and Saussereau [15] have studied the regularity in the sense of Malliavin Calculus of the solution of Equation (1.2), and they have established the absolute continuity of the law of the random variable X_t under some non-degeneracy conditions on the coefficient σ.

The main result of this paper is the flow and homeomorphic properties of X as a function of the initial condition x. Since the solution of (1.2) is defined path-wise, ordinary (i.e., deterministic) methods are in use here. Namely, we use the estimates found in [14] and approximate fractional Brownian motion by a sequence of regular processes to prove that the solution $\{X_{rt}(x),\, 0 \leq r \leq t \leq T,\, x \in \mathbb{R}^d\}$ of

$$X_{rt}(x) = x + \int_r^t \sigma\left(s, X_{rs}(x)\right) dB^H(s) + \int_r^t b(s, X_{rs}(x))ds,$$

defines a flow of $\mathbb{R}^d$-homeomorphisms.

The paper is organized as follows. Section 2 contains some preliminaries on fractional calculus. In Section 3 we review some results on the properties of the solution of Equation (1.2) and we establish some continuity estimates as a function of the initial condition and the driven input, which are needed later. finally in Section 4 we show that Equation (1.2) defines a flow of homeomorphisms.

2 Preliminaries

Let $a, b \in$, $a < b$. Let $f \in L^1(a,b)$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann-Liouville integrals of f of order α are defined for almost all $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, dy,$$

respectively, where $(-1)^{-\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^\infty r^{\alpha-1} e^{-r} dr$ is the Euler function. Let $I_{a+}^{\alpha}(L^p)$ (resp. $I_{b-}^{\alpha}(L^p)$) the image of $L^p(a,b)$ by the operator I_{a+}^{α} (resp. I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ (resp. $f \in I_{b-}^{\alpha}(L^p)$) and $0 < \alpha < 1$ then

the Weyl derivative

$$D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} dy \right) 1_{(a,b)}(x) \quad (2.3)$$

$$\left(\text{resp.} \quad D_{b-}^{\alpha} f(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{f(x)-f(y)}{(y-x)^{\alpha+1}} dy \right) 1_{(a,b)}(x) \right) \quad (2.4)$$

is defined for almost all $x \in (a,b)$.

For any $0 < \lambda \leq 1$, denote by $C^{\lambda}(0,T)$ the space of λ-Hölder continuous functions $f : [0,T] \to \mathbb{R}$, equipped with the norm $\|f\|_{\infty} + \|f\|_{\lambda}$, where

$$\|f\|_{\lambda} = \sup_{0 \leq s < t \leq T} \frac{|f(t)-f(s)|}{(t-s)^{\lambda}}.$$

Recall from [16] that we have:

- If $\alpha < \dfrac{1}{p}$ and $q = \dfrac{p}{1-\alpha p}$ then $I_{a+}^{\alpha}(L^p) = I_{b-}^{\alpha}(L^p) \subset L^q(a,b)$.
- If $\alpha > \dfrac{1}{p}$ then$I_{a+}^{\alpha}(L^p) \cup I_{b-}^{\alpha}(L^p) \subset C^{\alpha-\frac{1}{p}}(a,b)$.

The linear spaces $I_{a+}^{\alpha}(L^p)$ are Banach spaces with respect to the norms

$$\|f\|_{I_{a+}^{\alpha}(L^p)} = \|f\|_{L^p} + \left\|D_{a+}^{\alpha} f\right\|_{L^p} \sim \left\|D_{a+}^{\alpha} f\right\|_{L^p},$$

and the same is true for $I_{b-}^{\alpha}(L^p)$.

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, from the classical paper by Young [17], the Riemann-Stieltjes integral $\int_a^b f dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f dg$ in terms of fractional derivatives (see [18]).

Proposition 2.1 *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda+\mu > 1$. Let $\lambda > \alpha$ and $\mu > 1-\alpha$. Then the Riemann Stieltjes integral $\int_a^b f dg$ exists and it can be expressed as*

$$\int_a^b f dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) \, D_{b-}^{1-\alpha} g_{b-}(t) \, dt, \quad (2.5)$$

where $g_{b-}(t) = g(t) - g(b)$.

In [18] Zähle introduced a generalized Stieltjes integral of f with respect to g defined by the right-hand side of (2.5), assuming that f and g are functions such that $g(b-)$ exists, $f \in I_{a+}^{\alpha}(L^p)$ and $g_{b-} \in I_{b-}^{1-\alpha}(L^q)$ for some $p, q \geq 1$, $1/p + 1/q \leq 1$, $0 < \alpha < 1$.

Let $\alpha < \frac{1}{2}$ and $d \in \mathbb{N}^*$. Denote by $W_0^{\alpha,\infty}(0, T; \mathbb{R}^d)$ the space of measurable functions $f : [0, T] \to \mathbb{R}^d$ such that

$$\|f\|_{\alpha,\infty} := \sup_{t\in[0,T]} \left(|f(t)| + \int_0^t \frac{|f(t) - f(s)|}{(t-s)^{\alpha+1}} ds \right) < \infty.$$

We have, for all $0 < \varepsilon < \alpha$

$$C^{\alpha+\varepsilon}(0, T; \mathbb{R}^d) \subset W_0^{\alpha,\infty}(0, T; \mathbb{R}^d) \subset C^{\alpha-\varepsilon}(0, T; \mathbb{R}^d).$$

Denote by $W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^m)$ the space of measurable functions $g : [0, T] \to \mathbb{R}^m$ such that

$$\|g\|_{1-\alpha,\infty,T} := \sup_{0<s<t<T} \left(\frac{|g(t) - g(s)|}{(t-s)^{1-\alpha}} + \int_s^t \frac{|g(y) - g(s)|}{(y-s)^{2-\alpha}} dy \right) < \infty.$$

Clearly, for all $\varepsilon > 0$ we have

$$C^{1-\alpha+\varepsilon}(0, T; \mathbb{R}^m) \subset W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^m) \subset C^{1-\alpha}(0, T; \mathbb{R}^m). \tag{2.6}$$

Moreover, if g belongs to $W_T^{1-\alpha,\infty}(0, T; \mathbb{R}^m)$, its restriction to $(0, t)$ belongs to $I_{t-}^{1-\alpha}(L^\infty(0, t; \mathbb{R}^m))$ for all t and

$$\begin{aligned}\Lambda_\alpha(g) &:= \frac{1}{\Gamma(1-\alpha)} \sup_{0<s<t<T} \left| \left(D_{t-}^{1-\alpha} g_{t-} \right)(s) \right| \\ &\leq \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \|g\|_{1-\alpha,\infty,T} < \infty. \end{aligned} \tag{2.7}$$

The integral $\int_0^t f dg$ can be defined for all $t \in [0, T]$ if g belongs to $W_T^{1-\alpha,\infty}(0, T)$ and f satisfies

$$\|f\|_{\alpha,1} := \int_0^T \frac{|f(s)|}{s^\alpha} ds + \int_0^T \int_0^s \frac{|f(s) - f(y)|}{(s-y)^{\alpha+1}} dy\ ds < \infty.$$

Furthermore the following estimate holds

$$\left| \int_0^T f dg \right| \leq \Lambda_\alpha(g) \|f\|_{\alpha,1}.$$

3 Stochastic differential equations driven by a fBm

We are going to consider first the case of a deterministic equation. Let $0 < \alpha < \frac{1}{2}$ be fixed. Let $g \in W_T^{1-\alpha,\infty}(0,T;\mathbb{R}^m)$. Consider the deterministic differential equation on $\mathbb{R}^d$

$$\xi_t^i = x_0^i + \int_0^t b^i(s,\xi_s)ds + \sum_{j=1}^m \int_0^t \sigma^{i,j}(s,\xi_s)\, dg_s^j,\ \ t \in [0,T], \tag{3.8}$$

$i = 1,...,d$, where $x_0 \in \mathbb{R}^d$, and the coefficients $\sigma^{i,j}, b^i : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are measurable functions. Set $\sigma = \left(\sigma^{i,j}\right)_{d\times m}$, $b = \left(b^i\right)_{d\times 1}$ and for a matrix $A = \left(a^{i,j}\right)_{d\times m}$ and a vector $y = \left(y^i\right)_{d\times 1}$ denote $|A|^2 = \sum_{i,j} \left|a^{i,j}\right|^2$ and $|y|^2 = \sum_i \left|y^i\right|^2$.

Let us consider the following assumptions on the coefficients.

(H1) $\sigma(t,x)$ is differentiable in x, and there exist some constants $0 < \beta, \delta \le 1$, $M_1, M_2, M_3 > 0$ such that the following properties hold:

$$\begin{cases} i) \quad \text{Lipschitz continuity} \\ |\sigma(t,x) - \sigma(t,y)| \le M_1|x-y|, \quad \forall\, x \in \mathbb{R}^d,\ \forall\, t \in [0,T] \\ ii) \quad \text{Hölder continuity} \\ |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(t,y)| \le M_2|x-y|^\delta, \\ \qquad\qquad\qquad \forall\, |x|, |y| \in \mathbb{R}^d,\ \forall t \in [0,T], i = 1,\ldots,d, \\ iii) \quad \text{Hölder continuity in time} \\ |\sigma(t,x) - \sigma(s,x)| + |\partial_{x_i}\sigma(t,x) - \partial_{x_i}\sigma(s,x)| \le M_3|t-s|^\beta \\ \qquad\qquad\qquad \forall\, x \in \mathbb{R}^d,\ \forall\, t,s \in [0,T]. \end{cases}$$

(H2) There exists constants $L_1, L_2 > 0$ such that the following properties hold:

$$\begin{cases} i) \quad \text{Local Lipschitz continuity} \\ \quad |b(t,x) - b(t,y)| \le L_1|x-y|,\ \forall\, |x|, |y| \in \mathbb{R}^d,\ \forall t \in [0,T], \\ ii) \quad \text{Linear growth} \\ \quad |b(t,x)| \le L_2(1+|x|),\ \forall x \in \mathbb{R}^d,\ \forall t \in [0,T]. \end{cases}$$

Set

$$\alpha_0 = \min\left\{\frac{1}{2}, \beta, \frac{\delta}{1+\delta}\right\}.$$

The following existence and uniqueness result has been proved in [14].

Theorem 3.1 *Suppose that the coefficients $\sigma(t,x)$ and $b(t,x)$ satisfy assumptions (H1) and (H2). Then, if $\alpha < \alpha_0$ there exists a unique solution of Equation (3.8) in the space $C^{1-\alpha}\left(0,T;\ \mathbb{R}^d\right)$.*

Actually, these conditions can be slightly relaxed. For instance, the Hölder continuity of the partial derivatives of σ and the Hölder continuity of the coefficient b may hold only locally (see [14] for the details).

We now state two theorems which are consequences of the estimates found in [14].

For any $\lambda \geq 0$ we introduce the equivalent norm in the space $W_0^{\alpha,\infty}\left(0,T;\mathbb{R}^d\right)$ defined by

$$\|f\|_{\alpha,\lambda} = \sup_{t\in[0,T]} e^{-\lambda t}\left(|f(t)| + \int_0^t \frac{|f(t)-f(s)|}{(t-s)^{\alpha+1}}\ ds\right).$$

Theorem 3.2 *Let us denote $\xi_t(x_0)$ the solution of (3.8) at time t with initial condition x_0. Fix $R>1$. Then there exists a constant C such that for any x_0 and x_1 in the ball $B(0,\,R) = \{x\in\mathbb{R}^d,\ |x|\leq R\}$ and for any $\lambda > \left[R\exp\left(C\left(1+\Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right)\right]^{\frac{1}{1-2\alpha}}$ we have*

$$\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} \leq \left(1 - R\exp\left(C\left(1+\Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right)\lambda^{2\alpha-1}\right)^{-1}\ |x_0-x_1|.$$

Proof. It is proved in [14] that there exists a constant C_1 such that if $\lambda_0 = C_1\left(1+\Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}$ then for any initial condition x_0 in the ball of radius R we have

$$\|\xi(x_0)\|_{\alpha,\lambda_0} \leq 2\left(1+|x_0|\right) \leq 4R. \tag{3.9}$$

Given a function $f\in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ we define as in [14]

$$\begin{aligned} F_t^{(b)}(f) &= \int_0^t b(s,\ f(s))ds,\\ G_t^{(\sigma)}(g,f) &= \int_0^t \sigma(s,\ f(s))dg(s),\\ \Delta(f) &= \sup_{r\in[0,T]}\int_0^r \frac{|f(r)-f(s)|^\delta}{(r-s)^{\alpha+1}}ds. \end{aligned}$$

If $f,h \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ (see [14]) there exist constants C_2 and C_3 such that

$$\left\| F^{(b)}(f) - F^{(b)}(h) \right\|_{\alpha,\lambda} \leq C_2 \lambda^{\alpha-1} \left\| f-h \right\|_{\alpha,\lambda}, \tag{3.10}$$

$$\begin{aligned} \left\| G^{(\sigma)}(g,f) - G^{(\sigma)}(g,h) \right\|_{\alpha,\lambda} &\leq C_3 \Lambda_\alpha(g) \lambda^{2\alpha-1} \\ &\quad \times \left(1 + \Delta(f) + \Delta(h)\right) \left\| f-h \right\|_{\alpha,\lambda}, \end{aligned} \tag{3.11}$$

$$\left\| G^{(\sigma)}(g,f) \right\|_{\alpha,\lambda} \leq C_4 \Lambda_\alpha(g) \lambda^{2\alpha-1} \left(1 + \left\| f \right\|_{\alpha,\lambda}\right) \tag{3.12}$$

Also, if $f \in W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)$ and h is a bounded measurable function, then

$$\left\| F^{(b)}(h) \right\|_{1-\alpha} \leq C_5 (1 + \left\| h \right\|_\infty), \tag{3.13}$$

$$\left\| G^{(\sigma)}(g,f) \right\|_{1-\alpha} \leq C_6 \Lambda_\alpha(g) \left(1 + \left\| f \right\|_{\alpha,\infty}\right). \tag{3.14}$$

We have the following estimate

$$\begin{aligned} \Delta(\xi(x_0)) &= \sup_{r\in[0,T]} \int_0^r \frac{|\xi_r(x_0) - \xi_s(x_0)|^\delta}{(r-s)^{\alpha+1}}\, ds \\ &\leq \frac{T^{\delta-\alpha(1+\delta)}}{\delta - \alpha(1+\delta)} \left\| \xi(x_0) \right\|_{1-\alpha}, \end{aligned} \tag{3.15}$$

and using (3.13), (3.14), and (3.9) we obtain

$$\begin{aligned} \left\| \xi(x_0) \right\|_{1-\alpha} &\leq |x_0| + \left\| F^{(b)}(\xi(x_0)) \right\|_{1-\alpha} + \left\| G^{(\sigma)}(\xi(x_0)) \right\|_{1-\alpha} \\ &\leq |x_0| + C_5 \left(1 + \left\| \xi(x_0) \right\|_\infty\right) + \Lambda_\alpha(g) C_6 \left(1 + \left\| \xi(x_0) \right\|_{\alpha,\infty}\right) \\ &\leq C_7 e^{\lambda_0 T} (1 + |x_0|)(1 + \Lambda_\alpha(g)) \\ &\leq 2C_7 e^{\lambda_0 T} R (1 + \Lambda_\alpha(g)) \\ &= 2C_7 \exp\left(TC_1 \left(1 + \Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right) R(1 + \Lambda_\alpha(g)). \end{aligned}$$

Hence, from (3.15) we get

$$\Delta(\xi(x_0)) \leq C_8 \exp\left(TC_1 \left(1 + \Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right) R(1 + \Lambda_\alpha(g)). \tag{3.16}$$

Using (3.10), (3.11), and (3.16) we obtain for x_0 and x_1 in the ball of radius

R

$$
\begin{aligned}
\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} &\le |x_0-x_1| + \|F^{(b)}(\xi(x_0)) - F^{(b)}(\xi(x_1))\|_{\alpha,\lambda} \\
&+ \left\|G^{(\sigma)}(g,\xi(x_0)) - G^{(\sigma)}(g,\xi(x_1))\right\|_{\alpha,\lambda} \\
&\le |x_0-x_1| + C_1\lambda^{\alpha-1}\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} \\
&+ C_2\Lambda_\alpha(g)\lambda^{2\alpha-1}\left(1+\Delta(\xi(x_0))+\Delta(\xi(x_1))\right)\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} \\
&\le |x_0-x_1| + C_1\lambda^{\alpha-1}\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} \\
&+ C_2\Lambda_\alpha(g)\lambda^{2\alpha-1}\left(1+2C_8\exp\left(TC_1\left(1+\Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right)R(1+\Lambda_\alpha(g))\right) \\
&\times \|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda}.
\end{aligned}
$$

As a consequence,

$$
\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda} \le |x_0-x_1| + \lambda^{2\alpha-1}\left(\exp\left(C\left(1+\Lambda_\alpha(g)\right)^{\frac{1}{1-2\alpha}}\right)R\right)\|\xi(x_0)-\xi(x_1)\|_{\alpha,\lambda}
$$

for some constant C, which implies the result. ■

Theorem 3.3 *The map*

$$
\begin{aligned}
\xi \ : \ W_T^{1-\alpha,\infty}(0,\,T;\mathbb{R}^m) &\longrightarrow W_0^{\alpha,\infty}(0,\,T;\mathbb{R}^d) \\
g &\longmapsto \xi,
\end{aligned}
$$

where ξ is the solution of (3.8) with $x_0 \in B(0,\,R)$, is continuous. Namely, for $\lambda > 0$ large enough we have

$$
\|\xi(g)-\xi(h)\|_{\alpha,\lambda} \le \frac{C_1\lambda^{2\alpha-1}\|\xi(g)\|_{\alpha,\lambda}}{1-C_2\lambda^{2\alpha-1}\left(1+\Lambda_\alpha(h)\right)}\Lambda_\alpha(g-h).
$$

Proof. With the previous notations, we have

$$
\xi(g) = x_0 + F^{(b)}(\xi(g)) + G^{(\sigma)}(g,\xi(g))
$$

and

$$
\xi(h) = x_0 + F^{(b)}(\xi(h)) + G^{(\sigma)}(h,\,\xi(h)).
$$

Thus,

$$
\begin{aligned}
\xi(g)-\xi(h) &= F^{(b)}(\xi(g)) - F^{(b)}(\xi(h)) \\
&+ G^{(\sigma)}(g-h,\,\xi(g)) + \left(G^{(\sigma)}(h,\xi(g)) - G^{(\sigma)}(h,\,\xi(h))\right).
\end{aligned}
$$

According to (3.12), for λ sufficiently large,

$$\begin{aligned}\|\xi(g)-\xi(h)\|_{\alpha,\lambda} \leq {} & C_2\lambda^{\alpha-1}\|\xi(g)-\xi(h)\|_{\alpha,\lambda} \\ & + C_4\lambda^{2\alpha-1}(1+\Lambda_\alpha(g-h))\|\xi(g)\|_{\alpha,\lambda} \\ & + C_3\lambda^{2\alpha-1}\Lambda_\alpha(h)\|\xi(g)-\xi(h)\|_{\alpha,\lambda}.\end{aligned}$$

Hence the result. ■

Consider Equation (1.2) on $\mathbb{R}^d$, for $t \in [0,T]$, where X_0 is a d-dimensional random variable, and the coefficients $\sigma^{i,j}, b^i : \Omega \times [0,T] \times \mathbb{R}^d \to \mathbb{R}$ are measurable functions.

Theorem 3.4 *Suppose that X_0 is an $\mathbb{R}^d$-valued random variable, the coefficients $\sigma(t,x)$ and $b(t,x)$ satisfy assumptions (H1) and (H2), where the constants might depend on ω, with $\beta > 1-H$, $\delta > 1/H - 1$. Then if $\alpha \in (1-H, \alpha_0)$, then there exists a unique stochastic process*

$$X \in L^0\left(\Omega, \mathcal{F}, \mathbb{P}\,; W_0^{\alpha,\infty}(0,T;\mathbb{R}^d)\right)$$

solution of the stochastic equation (1.2) and, moreover, for $\mathbb{P}$-almost all $\omega \in \Omega$

$$X(\omega,\cdot) = \left(X^i(\omega,\cdot)\right)_{d\times 1} \in C^{1-\alpha}\left(0,T;\mathbb{R}^d\right).$$

Consider the particular case where $b=0$ and σ is time independent, that is,

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s^H. \tag{3.17}$$

By the above theorem this equation has a unique solution provided σ is continuously differentiable, and σ' is bounded and Hölder continuous of order $\delta > \frac{1}{H} - 1$.

In [6] Hu and Nualart have established the following estimate. Choose $\theta \in \left(\frac{1}{2}, H\right)$. Then, the solution to Equation (3.17) satisfies

$$\sup_{0\le t\le T}|X_t| \le 2^{1+kT(\|\sigma'\|_\infty \vee |\sigma(0)|)\|B^H\|_\theta^{1/\theta}}\left(|X_0|+1\right). \tag{3.18}$$

If σ is bounded and $\|\sigma'\| \neq 0$ this estimate can be improved in the following way

$$\sup_{0\le t\le T}|X_t| \le |X_0| + k\|\sigma\|_\infty\left(T^\theta\|B^H\|_\theta^{\frac{1}{\theta}} \vee T\|\sigma'\|_\infty^{\frac{1-\theta}{\theta}}\|B^H\|_\theta^{\frac{1}{\theta}}\right). \tag{3.19}$$

These estimates improve those obtained by Nualart and Răşcanu in [14] based on a suitable version of Gronwall's lemma. The estimates (3.18) and (3.19) lead to the following integrability properties for the solution of Equation (3.17).

Theorem 3.5 *Consider the stochastic differential equation (3.17), and assume that $E(|X_0|^p) < \infty$ for all $p \geq 2$. If σ' is bounded and Hölder continuous of order $\delta > \frac{1}{H} - 1$, then*

$$E\left(\sup_{0\leq t\leq T}|X_t|^p\right) < \infty \tag{3.20}$$

for all $p \geq 2$. If furthermore σ is bounded and $E\left(\exp(\lambda|X_0|^\gamma)\right) < \infty$ for any $\lambda > 0$ and $\gamma < 2H$, then

$$E\left(\exp\lambda\left(\sup_{0\leq t\leq T}|X_t|^\gamma\right)\right) < \infty \tag{3.21}$$

for any $\lambda > 0$ and $\gamma < 2H$.

In [15] Nualart and Saussereau have proved that the random variable X_t belongs locally to the space $\mathbb{D}^\infty$ if the function σ is infinitely differentiable and bounded together with all its partial derivatives. As a consequence, they have derived the absolute continuity of the law of X_t for any $t > 0$ assuming that the initial condition is constant and the vector space spanned by $\{(\sigma^{i,}(x_0))_{1\leq i\leq d}, 1 \leq j \leq m\}$ is $\mathbb{R}^d$.

Applying Theorem 3.5 Hu and Nualart have proved in [6] that if the function σ is infinitely differentiable and bounded together with all its partial derivatives, then for any $t \in [0, T]$ the random variable X_t belongs to the space $\mathbb{D}^\infty$. As a consequence, if the matrix $a(x) = \sigma\sigma^T(x)$ uniformly elliptic, then, for any $t > 0$ the probability law of X_t has an C^∞ density. In a recent paper, Baudoin and Coutin [1] have extended this result and derived the regularity of the density under Hörmander hypoellipticity conditions.

4 Flow of homeomorphisms

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be the uniform partition of the interval $[0, T]$. That is $t_k = \frac{kT}{n}$, $k = 0, \ldots, n$. We denote by $B^{n,H}$ the polygonal approximation of the fractional Brownian motion defined by

$$B_t^{n,H} = \sum_{k=0}^{n-1}\left(B_{t_k}^H + \frac{n}{T}(t - t_k)\left(B_{t_{k+1}}^H - B_{t_k}^H\right)\right)\mathbf{1}_{(t_k,t_{k+1}]}(t).$$

In order to get a precise rate for these approximations we will make use of the following exact modulus of continuity of the fractional Brownian motion. There exists a random variable G such that almost surely for any $s,t \in [0,T]$ we have

$$\left|B_t^H - B_s^H\right| \le G|t-s|^H \sqrt{\log\left(|t-s|^{-1}\right)}. \tag{4.22}$$

Fix $\theta < H$. We have the following result, which provides the rate of convergence of these approximations in Hölder norm.

Lemma 4.1 *There exist a random variable $C_{T,\beta}$ such that*

$$\|B^H - B^{n,H}\|_{C^\theta(0,T;\mathbb{R}^m)} \le C_{T,\beta} n^{\theta-H} \sqrt{\log n}. \tag{4.23}$$

Proof. To simplify the notation we will assume that $m=1$. Fix $0 < s < t < T$ and assume that $s \in [t_l, t_{l+1}]$ and $t \in [t_k, t_{k+1}]$. Let us first estimate

$$h_1(s,t) = \frac{1}{(t-s)^\theta}\left|B_t^{n,H} - B_t^H - (B_s^{n,H} - B_s^H)\right|.$$

If $t - s \ge \frac{T}{n}$, then using (4.22) we obtain

$$\begin{aligned}
|h_1(s,t)| &\le T^{-\beta} n^\beta \left[\left|B_{t_k}^H - B_t^H + \frac{n}{T}(t-t_k)\left(B_{t_{k+1}}^H - B_{t_k}^H\right)\right|\right. \\
&\quad \left. + \left|B_{t_l}^H - B_s^H + \frac{n}{T}(s-t_l)\left(B_{t_{l+1}}^H - B_{t_l}^H\right)\right|\right] \\
&\le 4GT^{-\theta+H} n^{-H+\theta}\sqrt{\log\left(n/T\right)}.
\end{aligned}$$

If $t - s < \frac{T}{n}$, then there are two cases. Suppose first that $s,t \in [t_k, t_{k+1}]$. In this case, if n is large enough we obtain using (4.22)

$$\begin{aligned}
|h_1(s,t)| &\le \frac{|B_t^H - B_s^H|}{(t-s)^\theta} + \frac{n}{T}\frac{|B_{t_{k+1}}^H - B_{t_k}^H|}{(t-s)^\theta}(t-s) \\
&\le G|t-s|^{H-\theta}\sqrt{\log|t-s|^{-1}} + GT^{-1+H}\sqrt{\log(n/T)}\, n^{1-H}(t-s)^{1-\theta} \\
&\le 2GT^{-\theta+H} n^{-H+\theta}\sqrt{\log\left(n/T\right)}.
\end{aligned}$$

On the other hand, if $s \in [t_{k-1}, t_k]$ and $t \in [t_k, t_{k+1}]$ we have, again if n is

large enough

$$
\begin{aligned}
|h_1(s,t)| \leq & \frac{1}{(t-s)^\theta} \left| B^H_{t_k} - B^H_t + \frac{n}{T}(t-t_k)\left(B^H_{t_{k+1}} - B^H_{t_k}\right) \right. \\
& \left. - \left\{ B^H_{t_k} - B^H_s - \frac{n}{T}(t_k - s)\left(B^H_{t_k} - B^H_{t_{k-1}}\right) \right\} \right| \\
\leq & \frac{1}{(t-s)^\theta} \left[|B^H_t - B^H_s| + \frac{n}{T}(t-s)\left(|B^H_{t_k} - B^H_{t_{k-1}}| + |B^H_{t_{k+1}} - B^H_{t_k}| \right) \right] \\
\leq & \frac{G}{(t-s)^\theta} \left[|t-s|^H \sqrt{\log|t-s|^{-1}} + 2(t-s)\left(\frac{n}{T}\right)^H \sqrt{\log(n/T)} \right] \\
\leq & \, 3GT^{-\theta+H} n^{-H+\theta} \sqrt{\log(n/T)}.
\end{aligned}
$$

This proves (4.23). ■

Corollary 4.1 *For any $\alpha \in (1-H, 1/2)$, we have:*

$$
\sup_n \Lambda_\alpha(B^{n,H}) < +\infty \text{ and } \lim_{n\to+\infty} \Lambda_\alpha(B^{n,H} - B^H) = 0.
$$

Proof. Choose $\eta > 0$ in such a way that $1 - \alpha + \eta < H$. According to (2.6) and (2.7), we have

$$
\Lambda_\alpha(B^{n,H}) \leq c_\eta \|B^{n,H}\|_{C^{1-\alpha+\eta}(0,T;\mathbb{R}^m)}
$$

and

$$
\Lambda_\alpha(B^{n,H} - B^H) \leq c_\eta \|B^{n,H} - B^H\|_{C^{1-\alpha+\eta}(0,T;\mathbb{R}^m)}.
$$

Then, Lemma 4.1 implies that the sequence $B^{n,H}$ converges to B^H in the norm of $C^{1-\alpha+\eta}(0,T;\mathbb{R}^m)$ which yields the results. ■

Consider for any $0 \leq r \leq t \leq T$ and any natural number $n \geq 1$ the following equations

$$
X^n_{rt}(x) = x + \int_r^t \sigma(s, X^n_{rs}(x))\, dB^{n,H}(s) + \int_r^t b(s, X^n_{rs}(x))ds, \tag{4.24}
$$

and

$$
Y^n_{rt}(x) = x + \int_r^t \sigma(s, Y^n_{st}(x))\, dB^{n,H}(s) + \int_r^t b(s, Y^n_{st}(x))ds. \tag{4.25}
$$

We know from standard results on ordinary differential equations that for any $n \geq 1$,

1. Equations (4.24) and (4.25) have a unique solution.

2. For any $x \in \mathbb{R}^d$, for any $0 \le r \le \tau \le t \le T$, $X^n_{\tau t}(X^n_{r\tau}(x)) = X^n_{rt}(x)$.

3. For any $x \in \mathbb{R}^d$, for any $0 \le r \le \tau \le t \le T$, $Y^n_{r\tau}(Y^n_{\tau t}(x)) = Y^n_{rt}(x)$.

4. The maps $(x \mapsto X^n_{rt}(x))$ and $(x \mapsto Y^n_{rt}(x))$ are $\mathbb{R}^d$-homeomorphisms inverse of each other:

$$X^n_{rt}(Y^n_{rt}(x)) = x \text{ and } Y^n_{rt}(X^n_{rt}(x)) = x.$$

We are then in position to prove our main theorem:

Theorem 4.1 *Assume that Hypothesis (H1) and (H2) hold. Then, claims 1, 2 3 and 4 also hold for the equations*

$$X_{rt}(x) = x + \int_r^t \sigma\left(s, X_{rs}(x)\right) dB^H(s) + \int_r^t b(s, X_{rs}(x))ds, \tag{4.26}$$

and

$$Y_{rt}(x) = x + \int_r^t \sigma\left(s, Y_{st}(x)\right) dB^H(s) + \int_r^t b(s, Y_{st}(x))ds. \tag{4.27}$$

Proof. Point 1 is proved in [14]. As to the second claim, proceed as follow:

$$\begin{aligned} X^n_{\tau t}(X^n_{r\tau}(x)) - X_{\tau t}(X_{r\tau}(x)) &= X^n_{\tau t}(X^n_{r\tau}(x)) - X^n_{\tau t}(X_{r\tau}(x)) \\ &\quad + (X^n_{\tau t} - X_{\tau t})(X_{r\tau}(x)). \end{aligned}$$

Fix $\varepsilon > 0$ and α such that $1 - H < \alpha < \frac{1}{2}$. Fix a trajectory $\omega \in \Omega$. Choose n_0 so that $\Lambda_\alpha(B^{n,H} - B^H) \le \varepsilon$ for all $n \ge n_0$ and choose λ such that $\lambda^{2\alpha-1} C_2 \sup_n \Lambda_\alpha(B^{n,H}) \le \frac{1}{2}$. Then, according to Theorem 3.3, for any $n \ge n_0$,

$$\|X^n_{r\cdot} - X_{r\cdot}\|_{\alpha,\lambda} \le \frac{C_1 \lambda^{2\alpha-1} \|X_{r\cdot}\|_{\alpha,\lambda}}{1 - C_2 \lambda^{2\alpha-1}\,(1 + \Lambda_\alpha(B^{n,H}))} \le 2C_1 \lambda^{2\alpha-1} \|X_{r\cdot}\|_{\alpha,\lambda} \varepsilon.$$

Hence, for $n \ge n_0$,

$$|(X^n_{\tau t} - X_{\tau t})(X_{r\tau}(x))| \le c\varepsilon.$$

The convergence of $X^n_{r\cdot}$ implies that there exists R such that for any $\tau \in [r, t]$ and for any $n \ge n_0$, $X^n_{r\tau}(x) \in B(0, R)$. Then, Theorem 3.2 implies that for

λ large enough

$$\begin{aligned}
&|X^n_{\tau t}(X^n_{r\tau}(x)) - X^n_{\tau t}(X_{r\tau}(x))| \\
&\quad \leq \left(1 - R\exp\left(C\left(1+\sup_n \Lambda_\alpha(B^{n,H})\right)^{\frac{1}{1-2\alpha}}\right)\lambda^{2\alpha-1}\right)^{-1} \\
&\qquad \times |X^n_{r\tau}(x) - X_{r\tau}(x)| \\
&\quad \leq c|X^n_{r\tau}(x) - X_{r\tau}(x)|.
\end{aligned}$$

We have thus proved that

$$\begin{aligned}
0 &= \lim_{n\to+\infty} X^n_{\tau t}(X^n_{r\tau}(x)) - X_{\tau t}(X_{r\tau}(x)) \\
&= \lim_{n\to+\infty} X^n_{rt}(x) - X_{\tau t}(X_{r\tau}(x)) \\
&= X_{rt}(x) - X_{\tau t}(X_{r\tau}(x)).
\end{aligned}$$

Other points are handled similarly. ■

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